



Oscillation Theorems for General Quasilinear Second-Order Difference Equations

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Abstract—The authors consider difference equations of the form

$$\Delta(a_n(\Delta y_n)^\alpha) + \phi(n, y_n, \Delta y_n) + q_n f(y_{n+1}) = 0, \quad (\text{E})$$

where $a_n > 0$, $q_n > 0$, f , and ϕ are continuous real valued functions, and $uf(u) > 0$ for $u \neq 0$. They give oscillation results for equation (E). Examples are included to illustrate the results. © 2001 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION

We consider the second-order quasilinear difference equation with a damping term

$$\Delta(a_n(\Delta y_n)^\alpha) + \phi(n, y_n, \Delta y_n) + q_n f(y_{n+1}) = 0, \quad (1)$$

where $n \in \mathbb{N} = \{0, 1, 2, \dots\}$, Δ is the forward difference operator $\Delta y_n = y_{n+1} - y_n$, $a_n > 0$, $q_n > 0$, $\alpha > 0$ is the ratio of odd positive integers, $\phi : \mathbb{N} \times \mathbb{R}^2 \rightarrow \mathbb{R}$ and $f : \mathbb{R} \rightarrow \mathbb{R}$ are real valued continuous functions, and $uf(u) > 0$ for $u \neq 0$. By a *solution* of equation (1), we mean a real sequence $\{y_n\}$ satisfying equation (1) for all $n \in \mathbb{N}$, and we say that the solution $\{y_n\}$ is *nonoscillatory* if it is eventually of constant sign, and it is *oscillatory* otherwise. An equation is said to be oscillatory if all its solutions are oscillatory.

In recent years, there has been an increasing interest in the study of the oscillatory and asymptotic properties of solutions of difference equations of type (1) and related equations, and we cite as recent contributions to this study the monographs of Agarwal *et al.* [1,2] and the papers of Chen and Zhang [3], Graef *et al.* [4], Li and Yeh [5], Thandapani *et al.* [6–13], Wong and Agarwal [14,15], and Zhang and Chen [16]. In particular, equations of the form

$$\Delta(a_n \Delta y_n) + p_n \Delta y_n + q_n f(y_{n+1}) = 0, \quad (\text{E}_1)$$

$$\Delta^2 y_n + p_n \phi(y_n, \Delta y_n) + q_n f(y_{n+1}) g(\Delta y_n) = 0, \quad (\text{E}_2)$$

and

$$\Delta(a_n \Delta y_n) + \phi(n, y_n, \Delta y_n) + q_n f(y_{n+1}) = 0, \quad (\text{E}_3)$$

are considered in the papers [6], [11], and [4], respectively, where the authors obtain criteria for the oscillation and asymptotic behavior of solutions of equations (E₁)–(E₃). In [14,15], the authors considered the equation

$$\Delta(a_n(\Delta y_n)^\alpha) + b_n(\Delta y_n)^\alpha + H(n, y_n, \Delta y_n) = 0, \quad (\text{E}_4)$$

and obtained conditions for the existence of positive monotone solutions as well as sufficient conditions for all solutions to oscillate.

In this paper, we obtain criteria for the oscillation and asymptotic behavior of solutions of equation (1) which include some of the results obtained for equations (E₁)–(E₄) as special cases, and extend and complement those in [4,6,7,11]. We will also assume that

- (H₁) there is a constant $M_1 > 0$ such that $f(u)/u^\alpha \geq M_1$ for all $u \neq 0$;
- (H₂) there are nonnegative real sequences $\{P_n\}$ and $\{p_n\}$ such that

$$-P_n v^{\alpha+1} \leq v \phi(n, u, v) \leq p_n v^{\alpha+1},$$

for all $(n, u, v) \in \mathbb{N} \times \mathbb{R} \times \mathbb{R}$ and $\phi(n, u, 0) = 0$.

As we see from (H₂), the damping term ϕ may take on positive and negative values and does not necessarily satisfy a condition of the form

$$v \phi(n, u, v) \geq 0, \quad \text{for } v \neq 0,$$

as is usually required by other authors. Notice that if ϕ does satisfy this inequality, then we can take the sequence $\{P_n\}$ in Condition (H₂) to be identically zero.

2. MAIN RESULTS

Our first two results ensure that any solution of equation (1) is either oscillatory or monotonically decays to zero.

THEOREM 1. *Suppose that (H₁), (H₂) hold and assume that for some $n_0 \in \mathbb{N}$,*

$$a_n - p_n > 0 \quad (2)$$

and

$$\{p_n\} \text{ is nonincreasing} \quad (3)$$

for all $n \geq n_0$. If

$$\sum_{n=n_0}^{\infty} \left(q_n - \frac{P_n}{M_1} \right) = \infty \quad (4)$$

and

$$\sum_{n=n_0}^{\infty} \left(\frac{1}{a_n} \sum_{s=n_0}^{n-1} q_s \right)^{1/\alpha} = \infty, \quad (5)$$

then every solution of equation (1) is either oscillatory or tends monotonically to zero as $n \rightarrow \infty$.

PROOF. Let $\{y_n\}$ be a nonoscillatory solution of equation (1), and without loss of generality, we assume that $y_n > 0$ for all $n \geq n_0$; the proof for the case where $y_n < 0$ eventually is similar and will be omitted. We consider the following three cases for the behavior of $\{\Delta y_n\}$.

CASE 1. $\{\Delta y_n\}$ is oscillatory.

- (a) First, suppose there exists an integer $n_1 \geq n_0$ such that $\Delta y_{n_1} < 0$. From equation (1), we have

$$\Delta(a_{n_1}(\Delta y_{n_1})^\alpha)\Delta y_{n_1} = -\phi(n_1, y_{n_1}, \Delta y_{n_1})\Delta y_{n_1} - q_{n_1}f(y_{n_1+1})\Delta y_{n_1},$$

so

$$\Delta y_{n_1}(a_{n_1+1}(\Delta y_{n_1+1})^\alpha - a_{n_1}(\Delta y_{n_1})^\alpha) > -p_{n_1}(\Delta y_{n_1})^{\alpha+1}.$$

Hence,

$$\Delta y_{n_1}(a_{n_1+1}(\Delta y_{n_1+1})^\alpha) > (a_{n_1} - p_{n_1})(\Delta y_{n_1})^{\alpha+1} > 0.$$

Thus, we have $\Delta y_{n_1+1} < 0$, and so by induction, we obtain $\Delta y_n < 0$ for all $n \geq n_1$.

- (b) Now suppose $\Delta y_{n_1} = 0$. Then, equation (1) implies $\Delta y_{n_1+1} < 0$ and we are back to Case (a). Thus, in either situation, we have $\Delta y_n < 0$ for all $n \geq n_1 + 1$. This contradicts the assumption that $\{\Delta y_n\}$ oscillates, so $\{\Delta y_n\}$ has fixed sign.

CASE 2. $\Delta y_n > 0$ for all $n \geq n_1 \geq n_0$. Define

$$z_n = \frac{a_n(\Delta y_n)^\alpha}{f(y_n)}; \quad (6)$$

then from equation (1), we have

$$\Delta z_n = -q_n - \frac{\phi(n, y_n, \Delta y_n)}{f(y_{n+1})} - \frac{a_n(\Delta y_n)^\alpha \Delta f(y_n)}{f(y_n)f(y_{n+1})}. \quad (7)$$

In view of Conditions (II₁) and (H₂), equation (7) implies

$$\Delta z_n \leq -q_n + \frac{P_n}{M_1} \left(1 - \frac{y_n}{y_{n+1}}\right)^\alpha,$$

so

$$\Delta z_n \leq -q_n + \frac{P_n}{M_1}.$$

Summing the last inequality from n_1 to n , we obtain

$$z_{n+1} \leq z_{n_1} - \sum_{s=n_1}^n \left(q_s - \frac{P_s}{M_1}\right).$$

Now condition (4) implies that $\{z_n\}$ is eventually negative, and this contradiction completes the proof in this case.

CASE 3. $\Delta y_n < 0$ for all $n \geq n_1 \geq n_0$. It follows that $\lim_{n \rightarrow \infty} y_n = b \geq 0$, and we claim that $b = 0$. Suppose $b > 0$ and define $u_n = a_n(\Delta y_n)^\alpha$ for $n \geq n_1$. Then, we have

$$\Delta u_n = -\phi(n, y_n, \Delta y_n) - q_n f(y_{n+1}).$$

Summing from n_1 to $n-1$ and using Condition (H₂), we obtain

$$u_n \leq u_{n_1} - \sum_{s=n_1}^{n-1} p_s(\Delta y_s)^\alpha - \sum_{s=n_1}^{n-1} q_s f(y_{s+1}). \quad (8)$$

In view of condition (3) and Abel's summation formula [1], equation (8) implies

$$u_n \leq u_{n_1} + p_{n_1} y_{n_1}^\alpha - f(y_{n+1}) \sum_{s=n_1}^{n-1} q_s + \sum_{s=n_1}^{n-1} \Delta f(y_s) \left(\sum_{t=n_1}^s q_t \right) \leq M - f(b) \sum_{s=n_1}^{n-1} q_s.$$

In view of condition (4), it is possible to choose an integer n_2 sufficiently large so that for $n \geq n_2$, we have

$$u_n \leq -\frac{f(b)}{2} \sum_{s=n_1}^{n-1} q_s$$

or

$$\Delta y_n \leq -\left(\frac{f(b)}{2}\right)^{1/\alpha} \left(\frac{1}{a_n} \sum_{s=n_1}^{n-1} q_s\right)^{1/\alpha}.$$

Summing the last inequality from n_2 to n , we obtain

$$y_{n+1} \leq y_{n_2} - \left(\frac{f(b)}{2}\right)^{1/\alpha} \sum_{s=n_2}^n \left(\frac{1}{a_s} \sum_{t=n_1}^{s-1} q_t\right)^{1/\alpha}.$$

Condition (5) implies that $\{y_n\}$ is eventually negative, which is a contradiction. This completes the proof of the theorem.

EXAMPLE 1. Consider the difference equation

$$\Delta \left(n(\Delta y_n)^3 \right) + \frac{2(\Delta y_n)^3}{2 + |\Delta y_n|} + 4(4n+1)y_{n+1}^3 = 0, \quad n \geq 2. \quad (e_1)$$

Here, $\alpha = 3$, $a_n = n$, $q_n = 4(4n+1)$, $\phi(n, u, v) = 2v^3/(2+|v|)$, $f(u) = u^3$, and we can take $P_n \equiv 0$ and $p_n \equiv 2$. All conditions of Theorem 1 are satisfied, so any solution of (e₁) is either oscillatory or tends to zero monotonically as $n \rightarrow \infty$. One such solution is $\{(-1)^n\}$.

Next, we discuss the oscillatory and asymptotic behavior of solutions of equation (1) with Condition (H₁) replaced by

(H₃) there exist constants $M_2 > 0$ and $\beta > \alpha$, with β the ratio of odd positive integers, such that $f(u)/u^\beta \geq M_2$ for all $u \neq 0$.

THEOREM 2. Suppose that in addition to (II₂), (II₃), conditions (2), (3), and (5) hold, $\{P_n\}$ is nonincreasing, and

$$\sum_{n=0}^{\infty} q_n = \infty. \quad (9)$$

Then every solution of equation (1) is either oscillatory or tends monotonically to zero as $n \rightarrow \infty$.

PROOF. Proceeding as in the proof of Theorem 1, we again see that condition (2) implies $\{\Delta y_n\}$ has fixed sign. Let $\Delta y_n > 0$ for all $n > n_1$ for some $n_1 \geq n_0$. Defining z_n as in Case 2 in the proof of Theorem 1, we have

$$\Delta z_n \leq -q_n - P_n \frac{(\Delta y_n)^\alpha}{M_2 y_{n+1}^\beta}.$$

Summing the last inequality from n_1 to n and using the monotonicity of $\{P_n\}$ and $\{y_n\}$, we obtain

$$z_{n+1} \leq z_{n_1} - \sum_{s=n_1}^n q_s + \frac{P_{n_1}}{M_2} \left(\int_{y_{n_1}}^{y_{n+1}} \frac{ds}{s^{\beta/\alpha}} \right)^\alpha \leq z_{n_1} - \sum_{s=n_1}^n q_s + \frac{P_{n_1}}{M_2} \left[\left(\frac{\alpha}{\beta - \alpha} \right) y_{n_1}^{\alpha - \beta/\alpha} \right]^\alpha,$$

since $\beta > \alpha$. Condition (9) implies that $\{z_n\}$ is eventually negative, which is a contradiction. The remainder of the proof is similar to that of Theorem 1 and will be omitted.

EXAMPLE 2. Consider the difference equation

$$\Delta(n^4(n+1)^2(\Delta y_n)^3) + \frac{n^3(n^4 + 2n^3 + n^2 + 1)(\Delta y_n)^5}{(n+2)(n+1)^7(1 + (\Delta y_n)^2)} + \frac{[(n+1)^9 + 1]}{(n+2)(n+1)^5} y_{n+1}^5 = 0, \quad n \geq 0. \quad (e_2)$$

We have $\alpha = 3$, $\beta = 5$, $a_n = n^4(n+1)^2$, $\phi(n, u, v) = [n^3(n^4 + 2n^3 + n^2 + 1)v^5]/(n+2)(n+1)^7(1 + v^2)$, $q_n = [(n+1)^9 + 1]/(n+2)(n+1)^5$, and $f(u) = u^5$. If we take $P_n \equiv 0$ and $p_n = [n^3(n^4 + 2n^3 + n^2 + 1)]/(n+2)(n+1)^7$, we see that all conditions of Theorem 2 are satisfied, and so any solution of (e_2) is either oscillatory or monotonically tends to zero as $n \rightarrow \infty$. One such solution is $\{1/n\}$.

In the next theorem, we obtain criteria for oscillation of all solutions of equation (1). To prove our result, we need the following lemma; it is a generalization of Lemma 2 in [11].

LEMMA 3. In addition to (H_2) and (2), assume that

$$\sum_{n=n_0}^{\infty} \left(\frac{1}{a_n} \prod_{s=n_0}^{n-1} \left(1 - \frac{p_s}{a_s} \right) \right)^{1/\alpha} = \infty. \quad (10)$$

If $\{y_n\}$ is a nonoscillatory solution of equation (1), then there is an integer $N \in \mathbb{N}$ such that $y_n \Delta y_n > 0$ for all $n \geq N$.

PROOF. Let $\{y_n\}$ be a nonoscillatory solution of equation (1), and assume $y_n > 0$ for $n \geq n_0 \in \mathbb{N}$. From the proof of Case 1 of Theorem 1, we have $\{\Delta y_n\}$ is eventually of fixed sign.

Let $\Delta y_n < 0$ for $n \geq N \geq n_0$. Defining $u_n = -a_n(\Delta y_n)^\alpha$, (1) and (H_2) yield

$$\Delta u_n + \frac{p_n}{a_n} u_n \geq 0. \quad (11)$$

Now (9) implies

$$u_n \geq u_N \prod_{s=N}^{n-1} \left(1 - \frac{p_s}{a_s} \right),$$

so

$$\Delta y_n \leq -u_N^{1/\alpha} \left(\frac{1}{a_n} \prod_{s=N}^{n-1} \left(1 - \frac{p_s}{a_s} \right) \right)^{1/\alpha}.$$

Summing from N to $n-1$, we have

$$y_n - y_N \leq a_N^{1/\alpha} \Delta y_N \sum_{s=N}^{n-1} \left(\frac{1}{a_s} \prod_{t=N}^{s-1} \left(1 - \frac{p_t}{a_t} \right) \right)^{1/\alpha},$$

for $n > N$. Condition (10) implies that $\{y_n\}$ is eventually negative, which is a contradiction. The proof in case $\{y_n\}$ is eventually negative is similar.

THEOREM 4. Assume that Conditions (H_1) , (H_2) , (2), (4), and (10) hold. Then every solution of equation (1) is oscillatory.

PROOF. The proof follows from Lemma 3 and the proof of Case 2 of Theorem 1.

The following result is a consequence of Lemma 3 and part of the proof of Theorem 2.

THEOREM 5. Assume Conditions (H_2) , (H_3) , (2), (9), and (10) hold, and $\{P_n\}$ is nonincreasing. Then every solution of equation (1) is oscillatory.

EXAMPLE 3. Consider the difference equation

$$\Delta((n+1)(\Delta y_n)^3) + \frac{2(\Delta y_n)^3}{(n+1)(2+|\Delta y_n|)} + \frac{4(4n^2+10n+5)}{(n+1)} y_{n+1}^3 = 0, \quad n \geq 1. \quad (e_3)$$

We have $a_n = (n+1)$, $\phi(n, u, v) = 2v^3/(n+1)(2+|v|)$, $q_n = 4(4n^2+10n+5)/(n+1)$, $f(u) = u^3$, and $\alpha = 3$, and we can take $P_n \equiv 0$ and $p_n = 1/(n+1)$. By Theorem 4, all solutions of (e_3) are oscillatory, and $\{(-1)^n\}$ is one such solution.

EXAMPLE 4. Consider the equation

$$\Delta((\Delta y_n)^3) - \frac{(\Delta y_n)^3}{(2n+1)[1+2|y_n|]} + \frac{2n+1+(2n+1)^3+(2n+3)^3}{(n+1)^3} y_{n+1}^3 = 0, \quad n \geq 1. \quad (e_4)$$

With $\alpha = 3$, $f(u) = u^3$, $P_n = 1/(2n+1)$, and $p_n \equiv 0$, all hypotheses of Theorem 4 are satisfied, so all solutions of (e_4) are oscillatory. Here, $\{(-1)^n\}$ is such a solution. Also note that this oscillatory solution is unbounded.

EXAMPLE 5. Consider the difference equation

$$\Delta((n+1)(\Delta y_n)^{1/5}) + \frac{(-1)^n(\Delta y_n)^{6/5}}{2+|\Delta y_n|} + 2^{-4/5}(4n+7)y_{n+1}^{1/3} = 0, \quad n \geq 1. \quad (e_5)$$

It is easy to see that all the hypotheses of Theorem 5 are satisfied with $\alpha = 1/5$, $\beta = 1/3$, $P_n \equiv 1$, and $p_n \equiv 1$; $\{y_n\} = \{(-1)^n\}$ is an oscillatory solution of (e_5) .

In the following theorem, we establish sufficient conditions for the oscillation of all solutions of equation (1) when condition (10) is not satisfied.

THEOREM 6. Assume (H_1) holds with $M_1 > 1$, Conditions (H_2) and (2) hold,

$$\rho_{n_0} = \sum_{n=n_0}^{\infty} \left(\frac{1}{a_n} \prod_{s=n_0}^{n-1} \left(1 - \frac{p_s}{a_s} \right) \right)^{1/\alpha} < \infty, \quad (12)$$

and

$$\liminf_{n \rightarrow \infty} q_n \rho_{n+1}^\alpha > \frac{1}{M_1}. \quad (13)$$

If there is a positive nonincreasing sequence $\{h_n\}$ such that

$$\sum_{n=n_0}^{\infty} (M_1 q_s - P_s) h_s = \infty, \quad (14)$$

then all solution of equation (1) are oscillatory.

PROOF. Let $\{y_n\}$ be a nonoscillatory solution of equation (1) and, without loss of generality, assume that $y_n > 0$ for all $n \geq n_0 \in \mathbb{N}$. The proof for the case $y_n < 0$ is similar and will be omitted. From the proof of Case 1 in the proof of Theorem 1, we have that $\{\Delta y_n\}$ eventually has fixed sign.

CASE 1. $\Delta y_n < 0$ for all $n \geq N$ for some $N \geq n_0$. From (1) and (H_2) , we have

$$\Delta u_n + \frac{p_n}{a_n} u_n \geq 0,$$

where $u_n = -a_n(\Delta y_n)^\alpha$. As in the proof of Lemma 3, we obtain

$$y_n - y_N \leq a_N^{1/\alpha} \Delta y_N \sum_{s=N}^{n-1} \left(\frac{1}{a_s} \prod_{t=N}^{s-1} \left(1 - \frac{p_t}{a_t} \right) \right)^{1/\alpha},$$

for $n \geq N$. Hence,

$$y_N \geq -(a_N(\Delta y_N)^\alpha)^{1/\alpha} \sum_{s=N}^{n-1} \left(\frac{1}{a_s} \prod_{t=N}^{s-1} \left(1 - \frac{p_t}{a_t} \right) \right)^{1/\alpha},$$

for $n \geq N$, and letting $n \rightarrow \infty$, we obtain

$$y_N^\alpha \geq -a_N(\Delta y_N)^\alpha \rho_N^\alpha. \quad (15)$$

Now (H_2) implies

$$\Delta(a_n(\Delta y_n)^\alpha) \Delta y_n + p_n(\Delta y_n)^{\alpha+1} + q_n f(y_{n+1}) \Delta y_n > 0,$$

or

$$a_{n+1}(\Delta y_{n+1})^\alpha - (a_n - p_n)(\Delta y_n)^\alpha + q_n f(y_{n+1}) < 0,$$

and so

$$a_{n+1}(\Delta y_{n+1})^\alpha + q_n f(y_{n+1}) < 0 \quad (16)$$

by (2). From (H_1) and (15), inequality (16) implies

$$a_{n+1}(\Delta y_{n+1})^\alpha - M_1 q_n a_{n-1}(\Delta y_{n-1})^\alpha \rho_{n+1}^\alpha < 0.$$

Thus,

$$q_n \rho_{n+1}^\alpha \leq \frac{1}{M_1},$$

for $n \geq N$, which contradicts (13).

CASE 2. $\Delta y_n > 0$ for all $n \geq N \geq n_0$. Defining

$$w_n = h_{n-1} a_n \frac{(\Delta y_n)^\alpha}{y_n^\alpha},$$

using the monotonicity of $\{h_n\}$ and Conditions (H_1) and (H_2) , we obtain

$$\Delta w_n \leq -M_1 h_n q_n + P_n h_n.$$

Summing from N to n , we have

$$w_{n+1} \leq w_N - \sum_{s=N}^n (M_1 q_s - P_s) h_s \rightarrow -\infty$$

as $n \rightarrow \infty$. This contradiction completes the proof of the theorem.

EXAMPLE 6. Consider the equation

$$\Delta(n(\Delta y_n)^5) + \frac{2(\Delta y_n)^7}{4 + (\Delta y_n)^2} + 32n(y_{n+1} + y_{n+1}^5) = 0, \quad n \geq 2. \quad (e_6)$$

We have $\alpha = 5$, $a_n = n$, $\phi(n, u, v,) = 2v^7/(4 + v^2)$, $q_n = 32n$, and $f(u) = u + u^5$. With $P_n \equiv 0$ and $p_n \equiv 1$, we have

$$\rho_n = \sum_{j=n}^{\infty} \left(\frac{1}{j} \prod_{i=n}^{j-1} \left(1 - \frac{1}{i} \right) \right)^{1/5} = \sum_{j=n}^{\infty} \left(\frac{n-1}{j(j-1)} \right)^{1/5} \leq (n-1)^{1/5} \left[\sum_{j=n}^{\infty} \Delta \left(-\frac{1}{j-1} \right) \right]^{1/5} < 1.$$

Now it is easy to see that all conditions of Theorem 6 with $h_n \equiv 1$ are satisfied, so every solution of (e_6) is oscillatory. One such solution is $\{y_n\} = \{(-1)^n\}$.

In conclusion, we just wish to point out that our results clearly hold if $(\Delta y_n)^\alpha$ in equation (1) is replaced by $|\Delta y_n|^\alpha \text{sgn } \Delta y_n$ so that the restriction on $\alpha > 0$ being the ratio of odd integers can be removed.

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